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A New Method for Fourier Analysis of Discontinuous and Peaky Periodic Waveforms

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Abstract—A simple algorithm is presented for calculating the Fourier coefficients of discontinuous and peaky periodic waveforms. The implementation of this algorithm requires only simple mathematical operations: summations and multiplications. Applications of the proposed algorithm are illustrated by two examples:

(1) a discontinuous waveform representing the output voltage of a cycloconverter, and
(2) a peaky function representing the radial flux density in the air gap of a hysteresis machine.

Keywords—Fourier coefficients, Waveforms, Algorithms.

1. INTRODUCTION

Periodic waveforms containing discontinuities and/or sharp peaks are widely encountered in physical and engineering problems. Such waveforms are usually obtained experimentally or numerically as the result of computer-based simulation programs. Analytical expressions of such waveforms are usually unknown. Thus, recourse to numerical techniques to find the Fourier coefficients from sampled data is inevitable. In this regard, the discrete Fourier transform (DFT) and the fast Fourier transform (FFT) are widely used. These techniques invariably demand a large number of equally-spaced sampled values in one period of the waveform under consideration. Obviously, for waveforms containing huge peaks, a huge number of samples is required in order to represent the fine details of the waveform near the peaks. This would require a huge number of multiplications to perform the FFT calculations. Moreover, the highest harmonic order obtainable using the DFT and FFT methods is usually limited by the sampling rate.

Alternatively, Fourier series coefficients can be obtained using the methods described in [1–6]. The method described in reference [1] does not require equidistant sampled data. Thus, it can be easily used for peaky waveforms. However, its application is restricted to periodic functions containing only odd harmonic terms. Moreover, this method assumes that a given periodic function can be divided into a number of parts; each part can be represented by a straight line or, for improved accuracy, by a second-order polynomial. Thus, a huge number of data points may be required to calculate Fourier coefficients of peaky waveforms with parts which may not easily lend

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themselves to a second-order polynomial. In such cases, recourse to straight-line representation may be inevitable. The method described in references [2–4] confines itself to the minimum by treating as sums the integrals involved in calculating the Fourier series coefficients. Although this approach results in short and simple calculations, the accuracy of the results is questionable especially for peaky waveforms. This is attributed to the assumption that in-between data points, the function is assumed constant. The method described in reference [5] requires numerical integration for evaluating the integrals involved in calculating the Fourier coefficients. While these integrals can be numerically evaluated using the trapezoidal rule, the accuracy of the results is highly questionable when applied to waveforms containing huge peaks, even when a large number of data points is used. Moreover, the procedure described in reference [5] requires equidistant data points. Finally, while the method proposed in reference [6] uses algebraic additions instead of integration to calculate the Fourier coefficients, its implementation requires values of the jumps of the function and its first derivative at the points of discontinuity. This is rarely available in most practical situations. Moreover, in the case of an unknown function to be analyzed or a function which is different from the types already analyzed, the method proposed in [6] is not applicable [7].

In this paper, the method proposed in [8] for calculating the Fourier coefficients of continuous periodic functions is modified so that it can be used for peaky and/or discontinuous periodic functions. Compared with the available methods, the proposed modified method has the following advantages.

(1) It requires only samples of the periodic function itself. Samples of the first derivative of the function are not required.
(2) It requires no numerical integration operations. Only algebraic additions are involved.
(3) It is applicable for peaky, discontinuous, and continuous periodic functions with even and/or odd symmetry; thus, it can yield both the odd and even harmonics of the function.
(4) Using a finite number of equidistant or unequidistant sampled data, there is no limitation on the number of calculated harmonics.
(5) The method can be directly applied to any set of sampled data, obtained by measurements or from computer simulation, and can be applied whether an analytical function representation is available or not.

Thus, the proposed method enjoys the advantages and avoids the disadvantages of available methods.

2. PROPOSED METHOD

In general, data obtained from computer-based simulation or experimental measurements, of a periodic peaky and/or discontinuous function $y(x)$, will be available in the form of a discrete number of N data points at x_n, y_n, $n = 1, 2, \dots, N$ at even (or odd) number of equally (or unequally) spaced intervals in one period; see Figure 1. Here, we propose to interpolate the function $y(x)$ between the data points by using $N-1$ straight line segments joined end to end as shown in Figure 1. The x-values of the segment joints are termed knots. The number of knots and their positions must generally be chosen so that closer knots are placed in regions where the function $y(x)$ is changing rapidly—peaky regions. The knots are not necessarily equally spaced. By denoting the slope of each segment by α_n, $n = 1, 2, \dots, N-1$, as shown in Figure 1, it is easy to show that the first derivative of the function $y(x)$ can be represented by Figure 2. Following the procedure described by Kreyszig [9], the jumps of the function $y(x)$ and its first derivative can be tabulated as shown in Tables 1 and 2. According to Kreyszig [9], the Fourier-series coefficients of equation (1)

$$y(x) = \delta_0 + \sum_{m=1}^{M} \left(\delta_m \cos\left(\frac{2m\pi}{B}x\right) + \gamma_m \sin\left(\frac{2m\pi}{B}x\right) \right) \tag{1}$$

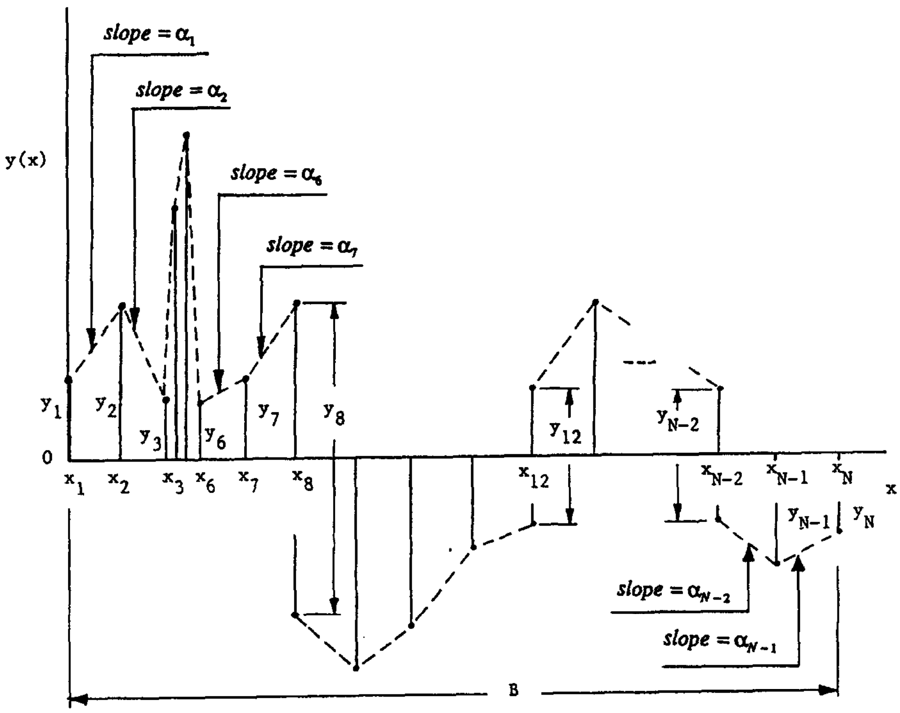

Figure 1. Measured or simulated data over one period B.

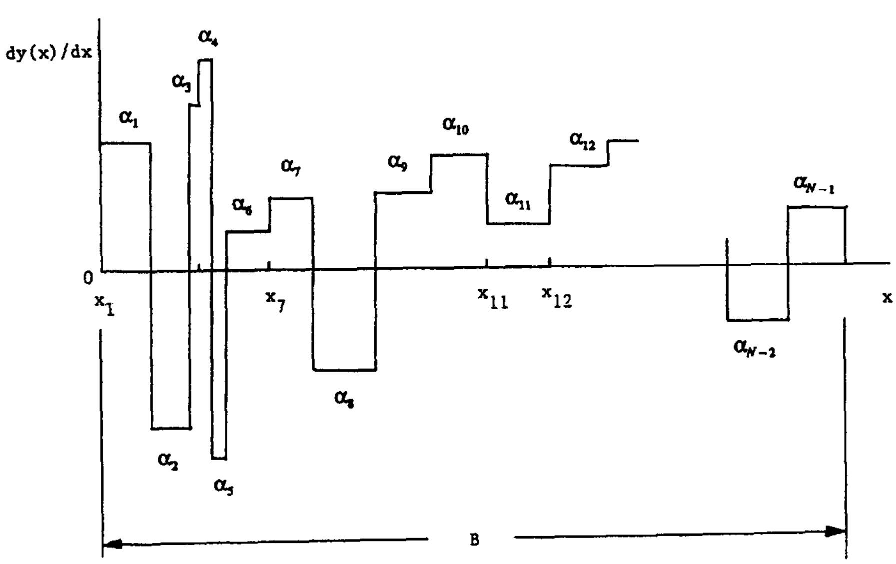

Figure 2. The first derivative of the function of Figure 1 obtained by interpolating the function $y(x)$ in-between data points by straight lines.

can be expressed as

$$\delta_m = \frac{1}{m\pi}\left(-\sum_{s=1}^{q} J_s \sin K_m x_s - \frac{1}{K_m}\sum_{s=1}^{p} J'_s \cos K_m x_s\right), \tag{2}$$

and

$$\gamma_m = \frac{1}{m\pi}\left(\sum_{s=1}^{q} J_s \cos K_m x_s - \frac{1}{K_m}\sum_{s=1}^{p} J'_s \sin K_m x_s\right), \tag{3}$$

Table 1. Jumps of the function $y(x)$ of Figure 1.

x_{js}	y_{js}
x_1	$+y_1$
x_8	$-y_8$
x_{12}	$+y_{12}$
$\vdots$	$\vdots$
x_{N-2}	$-y_{N-2}$
x_N	$+y_N$

Table 2. Jumps of the function $y'(x)$ of Figure 2.

x_{js}	y'_{js}
x_1	α_1
x_2	$\alpha_2 - \alpha_1$
x_3	$\alpha_3 - \alpha_2$
$\vdots$	$\vdots$
x_{N-1}	$\alpha_{N-1} - \alpha_{N-2}$
x_N	α_{N-1}

where $K_m = 2m\pi/B$, B is the period of the periodic function, q is the number of jumps of the function $y(x)$, p is the number of jumps of the first derivative of the function $y(x)$, J_s is the value of the jump of the function $y(x)$ at x_s, and J'_s is the value of the jump of the first derivative of the function $y(x)$ at x_s. Equations (2) and (3) are modified versions of equations (11a) and (11b) of Kreyszig [9] with period B rather than 2π. Thus, using Tables 1 and 2 and equations (2) and (3), the Fourier coefficients of the function of Figure 1 can be expressed as

$$\begin{aligned}\delta_m &= \frac{-1}{m\pi}\sum_{s=1}^{q} y_{js}\sin\left(\frac{2m\pi}{B}x_{js}\right)\\ &+\frac{-B}{2(m\pi)^2}\left(\alpha_1-\alpha_{N-1}+\sum_{s=1}^{N-2}(\alpha_{s+1}-\alpha_s)\cos\left(\frac{2m\pi}{B}x_{s+1}\right)\right),\end{aligned}\tag{4}$$

and

$$\gamma_m = \frac{1}{m\pi}\sum_{s=1}^{K} y_{js}\cos\left(\frac{2m\pi}{B}x_{js}\right)+\frac{-B}{2(m\pi)^2}\left(\sum_{s=1}^{N-2}(\alpha_{s+1}-\alpha_s)\sin\left(\frac{2m\pi}{B}x_{s+1}\right)\right),\tag{5}$$

where y_{js}, $s = 1, 2, \ldots, p$, is the value of the jump of $y(x)$ at x_{js}, $s = 1, 2, \ldots, p$. An approximate value of δ_0, if required, can be obtained by averaging the samples of $y(x)$ over the period B [4].

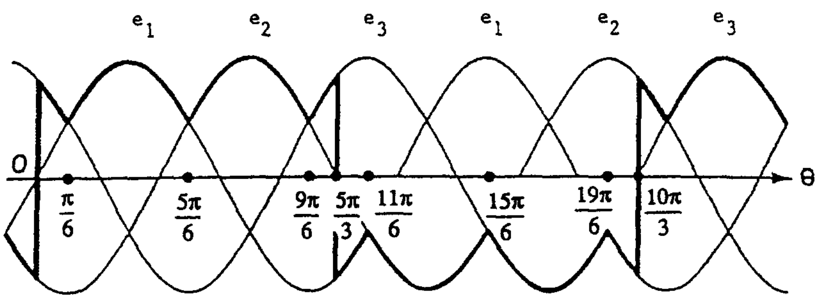

Figure 3. Output voltage waveform of a cycloconverter using square-wave modulation [6].

Table 3. Harmonic analysis of the cycloconverter output voltage waveform of Figure 3. $\delta_m = 0$, $m = 1, 2, 3, \ldots$, $\gamma_m = 0$, $m = 2, 4, 6, \ldots$, $\delta_0 = 0$, and $B = 10\pi/3$.

Order of Harmonic m	γ_m
1	1.0508803
3	0.3485996
5	−0.0000034
7	0.1422119
9	0.0887248
11	0.1275668
13	0.0929361
15	0.0990977
17	0.0693584
19	0.0666883
21	0.0459839
⋮	⋮
201	0.0054120
⋮	⋮
401	0.0027494

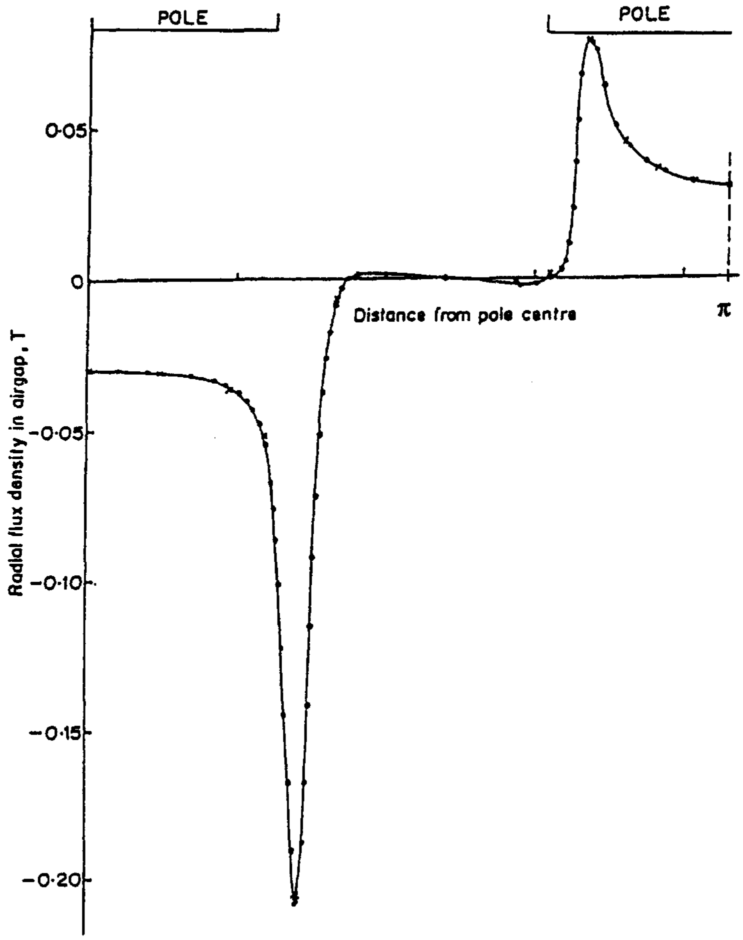

Figure 4. Radial flux density variation in the air gap of a hysteresis machine [10].

Table 4. Harmonic analysis of the peaky waveform of Figure 4. $B = 2\pi$, $\delta_m = \gamma_m = 0$, $m = 2, 4, 6, \ldots$, and $\delta_0 = 0$.

Order of Harmonic m	δ_m	γ_m
1	−0.0463	−0.0185
3	0.0167	0.0020
5	0.0044	0.0230
7	−0.0092	−0.0185
9	0.0108	−0.0079
11	−0.0072	0.0173
13	−0.0061	−0.0059
15	0.0139	−0.0045
17	−0.0042	0.0064
19	−0.0083	−0.0054
21	0.0074	0.0007
$\vdots$	$\vdots$	$\vdots$
201	0.000008	−0.000017
$\vdots$	$\vdots$	$\vdots$
401	−0.000003	−0.000018

Table 5. Sample of the measured and calculated data points of the peaky waveform of Figure 4.

Distance from Pole Center (Radians)	Radial Flux Density (Measured), T	Radial Flux Density (Calculated), T
0.143661	−0.02975	−0.02975
0.513086	−0.03158	−0.03159
0.687534	−0.03509	−0.03512
0.790151	−0.04070	−0.04073
0.974812	−0.14414	−0.14321
1.026172	−0.20772 (peak)	−0.20697 (peak)
1.241663	−0.00351	−0.00357
2.196004	−0.00182	−0.00185
2.442283	0.07789 (peak)	0.07755 (peak)
2.821964	0.03512	0.03513
3.140000	0.02951	0.02952

From equations (4),(5) it is obvious that calculations of δ_m and γ_m for any value of m requires only simple mathematical operations. Also, inspection of equations (4) and (5) suggests that as m becomes infinite, the Fourier-series coefficients δ_m and γ_m always approach zero.

Using equations (4),(5), any set of data points, with or without discontinuities, obtained by measurement or computer-based simulation, at any sequence of intervals, can be represented by a Fourier series to any degree of accuracy by using as many terms as necessary. The number of δ_m and γ_m terms is allowed to increase until the inclusion of the next term is seen to make a negligible contribution towards a best fit criteria—the minimum relative root-mean-square error.

3. EXAMPLES

To illustrate the accuracy of the proposed algorithm, consider the following examples.

EXAMPLE 1. Discontinuous waveform. Figure 3 shows the output voltage waveform of a cycloconverter using square-wave modulation [6]. Samples can be directly obtained from Figure 3, and using equations (4) and (5), values of δ_m and γ_m were calculated and the results are tabulated in Table 3. As expected, only odd-order sine components are present in this waveform. To check the validity of this analysis, the waveform was reconstructed using the Fourier coefficients of Table 3. Using 200 coefficients, a relative root-mean-square error of 0.2% was achieved.

EXAMPLE 2. Peaky waveform. Figure 4 shows the waveform of the radial flux density variation in the air gap of a hysteresis machine [10]. Calculated values of δ_m and γ_m are shown in Table 4. As expected, only odd-order sine and cosine components are present in this waveform. To check the validity of this analysis, the waveform was reconstructed using the Fourier coefficients of Table 4. Table 5 shows a sample of the results from which it can be concluded that high accuracy is achieved. From Table 5, it can be seen that peaks can be calculated with a relative root-mean-square error less than 0.5%.

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